Speedup in Quantum Adiabatic Evolution Algorithm

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Z. Naturforsch. **58a,** 201 – 203 (2003); received March 17, 2003

The quantum adiabatic evolution algorithm suggested by Farhi et al. was effective in solving NP-complete problems. The algorithm is governed by the adiabatic theorem. Therefore, in order to reduce the running time, it is essential to examine the minimum energy gap between the ground level and the next one in the evolution. In this paper we show a way of speedup in the quantum adiabatic evolution algorithm, using an extended Hamiltonian. We present the exact relation between the energy gap and the elements of the extended Hamiltonian, which provides a new point of view to reduce the running time.

Key words: Quantum Computation; Quantum Search; Quantum Adiabatic Evolution.

It is generally expected that quantum computation can solve problems which are hard to figure out classically. Shor's quantum factorization algorithm [1] and Grover's quantum search algorithm [2] encourage such expectation. The problem to factorize a large number needs an exponential time dependence when a known classical algorithm is applied. Shor's algorithm, however, solves the problem with an exponential speedup. Grover's algorithm finds a target with quadratic speedup, compared with any known classical search algorithm.

The quantum adiabatic evolution algorithm was proposed by Farhi et al. and applied to solve NP-complete problems [3 4]. The algorithm is governed by the adiabatic theorem, which indicates that the total evolution time is proportional to the inverse square of the minimum difference between the lowest energy level and the others during the adiabatic evolution. Therefore, the energy difference between the two levels is crucial to reduce the time needed for the algorithm.

In this paper we consider the extended Hamiltonian for the quantum adiabatic evolution algorithm in order to understand what factors in the extended Hamiltonian are related to the minimum energy gap. We also present the exact relation between the minumum energy gap and the elements of the extended Hamiltonian and show some ways to speedup the algorithm.

We first review the quantum adiabatic evolution algorithm. Suppose that a problem of finding an assignment is given. The quantum adiabatic evolution algorithm provides the assignment by an evolution of the Hamiltonian H, which is appropriately constructed to solve the given problem. The assignment corresponds to the lowest energy level of the Hamiltonian after the evolution. The algorithm consists of three steps: 1.) Preparation of a superposition of all states in the lowest energy level of the Hamiltonian(initialization), 2.) Adiabatic evolution of the Hamiltonian, and 3.) Readout. Notice that the lowest energy level at each time of the adiabatic evolution is the development of the assignment. This means that, if the lowest energy level crosses others during the Hamiltonian evolution, the algorithm would fail. Therefore the change rate of evolution of the system should be slow, so that energy level-crossing does not occur. Then the following question arises: how slow should the change rate be to keep the system in the ground state? The answer comes from the adiabatic theorem, which implies that the probability to obtain the assignment after the evolution is $(1-\varepsilon^2)$, subject to

$$|\langle \frac{\mathrm{d}H}{\mathrm{d}t} \rangle| / g_{\min}^2 \leq \varepsilon, \tag{1}$$

where $g_{\min}$ is the minimum energy gap between the lowest two energy eigenvalues during the Hamiltonian evolution.

Consider the quantum adiabatic Hamiltonian of Fahri et al. [4] Since the initial and the final state, $|\psi\rangle$ and $|\alpha\rangle$, are encoded in the lowest energy levels of the initial and final Hamiltonians, H_{i} and H_{f} respectively, they are usually given as [3]

$$H_{\mathrm{i}} = 1 - |\psi\rangle\langle\psi|, \tag{2}$$

0932–0784 / 03 / 0400–0201 $ 06.00

$$H_{\mathrm{f}} = 1 - |\alpha\rangle\langle\alpha|. \tag{3}$$

Then the full Hamiltonian is

$$H(s(t)) = (1 - s(t))H_{\mathrm{i}} + s(t)H_{\mathrm{f}},$$

where $s(t)$ is a homotopy from H_{i} to H_{f} to apply the adiabatic evolution. If the homotopy is given as a constant function $s(t) = t/T$, where t is the time and T the total evoluton time, then it is said that the adiabatic evolution is globally applied [3]. The local adiabatic evolution means that the function $s(t)$ depends on the energy gap at each time $t \in [0,T]$. Roland and Cerf [5] showed that the local adiabatic evolution provides a quadratic speedup in finding a target in an unstructured database.

Let us consider an extended Hamiltonian for the quantum adiabatic evolution algorithm. For the general description of the Hamiltonian we start from

$$H = k_1 + k_2|\psi\rangle\langle\psi| + k_3|\alpha\rangle\langle\alpha| + k_4|\alpha\rangle\langle\psi| + k_5|\psi\rangle\langle\alpha|,$$

where k_i $(i = 1,2,3,4,5)$ is complex, $|\alpha\rangle$ is the ground state and $|\psi\rangle$ the initial state that is composed of superposed N states. This Hamiltonian has every element about the initial and final states. Since the observable H is hermitian, we have

$$H = k_1 + k_2|\psi\rangle\langle\psi| + k_3|\alpha\rangle\langle\alpha| + k_4(|\alpha\rangle\langle\psi| + |\psi\rangle\langle\alpha|)$$

with k_i $(i = 1,2,3,4)$ in units of energy.

The initial and final Hamiltonians are

$$H_{\mathrm{i}} = a_1 + a_2|\psi\rangle\langle\psi| + a_3|\alpha\rangle\langle\alpha| + a_4(|\alpha\rangle\langle\psi| + |\psi\rangle\langle\alpha|),$$

$$H_{\mathrm{f}} = b_1 + b_2|\psi\rangle\langle\psi| + b_3|\alpha\rangle\langle\alpha| + b_4(|\alpha\rangle\langle\psi| + |\psi\rangle\langle\alpha|),$$

where a_i and b_i $(i = 1,2,3,4)$ are constants in unit of energy. The full Hamiltonian is then

$$H(s) = (1 - s)H_i + sH_f$$

We are going to refine the Hamiltonian H to perform the quantum adiabatic evolution algorithm. Note that the constraints for an adiabatic evolution Hamiltonian are

1.) The ground state of the initial Hamiltonian(the initial state which is easy to prepare) is a superposition of all states.

2.) The ground state of the final Hamiltonian is the final state(assignment).

3.) There is no level crossing between the ground state and the others at $t \in [0,T]$.

By the first constraint, the ground state of the initial Hamiltonian should be the initial state $|\psi\rangle$. This provides

$$a_4 = -a_3 x, \tag{4}$$

$$a_3 > a_2, \tag{5}$$

where $x = \langle\alpha|\psi\rangle$. Moreover, the lowest energy value of H_{i} is $a_1 + a_2 - a_3 x^2$, which should be nonnegative. The second constraint, that the ground state of H_{f} is to be $|\alpha\rangle$, gives

$$b_4 = -b_2 x, \tag{6}$$

$$b_2 > b_3. \tag{7}$$

In addition, we know that the lowest energy value of H_{f} is $b_1 + b_3 - b_2 x^2$, which should be nonnegative. The third constraint holds by the homotopy $s(t)$.

We now apply the full Hamiltonian $H = (1-s)H_{\mathrm{i}} + sH_{\mathrm{f}}$ to the quantum adiabatic evolution algorithm. Then the energy gap $g^2(s)$ between the lowest energy level and the next one is

$$g^2(s) = As^2 + Bs + C, \tag{8}$$

$$A = (a_2 - a_3)^2 + 2(-1 + 2x^2)(a_2 - a_3)(b_2 - b_3) + (b_2 - b_3)^2,$$

$$B = -2(a_2 - a_3)(a_2 - a_3 + (-1 + 2x^2)(b_2 - b_3)),$$

$$C = (a_2 - a_3)^2. \tag{9}$$

Through some calculation the following conditions are obtained :

1. A is positive.
2. B is negative.
3. $-B/2A$ is in the interval $(0,1)$.
4. $g^2(s=0) = (a_2 - a_3)^2$ and $g^2(s=1) = (b_2 - b_3)^2$.

These are easy to check. They imply that $g^2(s)$ is a parabola which has its minimum at $s = -B/2A \in (0,1)$. The minimum energy gap at $s = -B/2A$ is

$$g^2_{\min} = \frac{4x^2(1-x^2)}{(\frac{1}{a})^2 + 2(1-2x^2)\frac{1}{ab} + (\frac{1}{b})^2}, \tag{10}$$

where $a = (a_3 - a_2)$ and $b = (b_2 - b_3)$. From the Cauchy-Schwarz inequality, $g^2_{\min}$ is minimal when $a = b$:

$$g^2_{\min} \geq a^2 x^2 \approx \frac{a^2}{N}.$$

The lower bound of the minimum gap is a^2/N.

Now consider the running time of the quantum adiabatic evolution algorithm with the extended Hamlitonan. We first examine the running time when the adiabatic passage is applied globally. In this case, the homotopy from $H_{\rm i}$ to $H_{\rm f}$ is given as $s(t) = t/T$. Then, from (1) we have

$$T \geq \frac{\sqrt{1-x^2}}{\varepsilon}\frac{1}{ax^2} \approx \frac{1}{\varepsilon}\frac{N}{a}.$$

The minimal time resource for the global adiabatic evolution is proportional to N/a. That is, the minimum energy gap can be manipulated with various values of a. In particular, the quardratic speedup $T = O(\sqrt{N})$ and the constant time resource $T = O(1)$ are obtained by choosing $a = N^{1/2}$ and $a = N$, respectively.

Recently, Roland and Cerf proposed a local adiabatic evolution to reduce the total evolution time [5]. 'Local' means that the change rate, $s(t)$, is not constant but proportional to the inverse square of the energy gap at each time. The total evolution time is obtained by the integration

$$T \geq \frac{1}{\varepsilon}\int_0^1 \mathrm{d}s \frac{|\langle \frac{\mathrm{d}H}{\mathrm{d}s}\rangle|}{g^2(s)}.$$

To minimize the evolution time T, the following proposition is presented.

Proposition 1 (Minimum Area) *Let $f(x) = a(x-p)^2 + c$ be a parabola defined in the interval $I = [0,1]$, where $a > 0$ and p and c are real. The integration*

$$\int_0^1 f(x)\mathrm{d}x$$

has the minimum value when $p = \frac{1}{2}$.

The proof is trivial. The proposition implies that $a = (a_3 - a_2) = (b_2 - b_3) = b$ is necessary to the optimization when the local adiabatic evolution is applied. We have then the running time

$$T \geq \frac{1}{\varepsilon\sqrt{1-x^2}}\frac{1}{ax} = O(\frac{1}{ax}) \approx O(\frac{\sqrt{N}}{a}).$$

The minimal time resource is proportional to $\sqrt{N}/a$. Similar to the case of the global adiabatic evolution, various choices of a provide the maximal speedup for the local adiabatic evolution. In particular, we have the exponential speedup $T = O(N^{1/4})$ for $a = N^{1/4}$, and the constant time resource $T = O(1)$ for $a = N^{1/2}$. The large value of a means to initialize the extended Hamiltonian with a large amount of energy.

Hence we have shown some ways for speedup in the quantum adiabatic evolution algorithms (e. g. in cases of local and global adiabatic evolutions), by considering the extended Hamiltonian. Furthermore, we have verified that the minimum energy gap depends only on the values $a = a_3 - a_2$ and $b = b_2 - b_3$ of the extended Hamiltonian. The change rate of the global adiabatic evolution depends only on the minimum energy gap, and the time resource of the local adiabatic evolution relies on two factors, which are the number of states and the place the minimum energy gap appears at. We have obtained that the condition $a = b$ is necessary to minimize the running time, and a large value of a provides a speedup in the quantum adiabatic evolution algorithm. This implies that a large value of a_3, which is the energy of the target state $|\alpha\rangle\langle\alpha|$ in the initial Hamiltonian, is crucial for the speedup, and so is a large value of b_2, which is the energy of the initial state $|\psi\rangle\langle\psi|$ in the final Hamiltonian. Note that the values a_1 and b_1 in the extended Hamiltonian are unrelated to the energy gap.

Acknowledgement

J. B. is supported in part by the Hanyang University and Y. K. is supported in part by the Fund of Hanyang University.

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